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Introduction to Particle Accelerator Physics

Solutions to Exercise 1

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1. Relativistic Particles

a) Recall from the lecture the definition of γ:

$$\gamma = \frac{E_{tot}}{E_0} = \frac{E_0 + E_{kin}}{E_0} = 1 + \frac{E_{kin}}{m_0 c^2}$$

The beam is accelerated up to $E_{kin} = 2.4$ GeV, so

$$\gamma = 1 + \frac{E_{kin}}{E_0} = 1 + \frac{2.4 \text{ GeV}}{511 \frac{\text{keV}}{\text{c}^2} \cdot c^2} = 1 + \frac{2.4 \cdot 10^9 \text{ eV}}{511 \cdot 10^3 \text{ eV}} \approx 4698$$

b)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \Longrightarrow \quad \beta = \sqrt{1-\gamma^{-2}} = 0.99999997733... \approx 1$$

$\Longrightarrow$ The electrons are highly relativistic: $v = \beta c \approx c$

c)

$$f_{rev} = \frac{v}{C} = \frac{\beta c}{C} \approx \frac{c}{C} \approx \frac{3 \cdot 10^8 \text{ m/s}}{288 \text{ m}} \approx 1.042 \text{ MHz}$$

d)

$$m_e = 511 \frac{\text{keV}}{c^2} = 511 \cdot 10^3 \text{ V} \cdot 1.602 \cdot 10^{-19} \text{C} \cdot \frac{1}{(3 \cdot 10^8 \text{ m/s})^2} = 9.11 \cdot 10^{-31} \text{ kg} = m_e$$

e)

$$\gamma = \frac{E_{tot}}{E_0} = \frac{E_0 + E_{kin}}{E_0} = 1 + \frac{E_{kin}}{m_0 c^2}$$

At SLS electrons are accelerated to 2.4 GeV: $\gamma_{SLS} = 1 + \frac{2.4 \text{ GeV}}{511 \text{ keV}} \approx 4698$
At LHC protons are accelerated to 7 TeV: $\gamma_{LHC} = 1 + \frac{7 \text{ TeV}}{938 \text{ MeV}} \approx 7464$
$\Longrightarrow$ The LHC protons have a higher kinetic energy than the SLS electrons.
Since $m_p = 1836 \cdot m_e$, protons at an energy of $1836 \cdot 2.4$ GeV ≈ 4.4 TeV have the same kinetic energy as the SLS' electrons.

2. Electron Beam in a Storage Ring

a) Recall from the lecture the definition of the magnetic rigidity:

$$B\rho = \frac{p}{e}$$

In more practical units this becomes:

$$B\rho\,[\mathrm{T}\cdot\mathrm{m}] = \frac{1}{0.29979}\cdot p\,[\mathrm{GeV/c}]$$

Here we assume the whole storage ring consists only of bending magnets, so $R = C/2\pi$ and we get:

$$\overline{B}R\,[\mathrm{T}\cdot\mathrm{m}] = \frac{1}{0.29979}\cdot p\,[\mathrm{GeV/c}] \quad\Longrightarrow\quad \overline{B}\,[\mathrm{T}] = \frac{1}{0.29979}\cdot p\,[\mathrm{GeV/c}]\cdot\frac{2\pi}{C\,[\mathrm{m}]} = 175\ \mathrm{mT}$$

b)

$$B\rho\,[\mathrm{T}\cdot\mathrm{m}] = \frac{1}{0.29979}\cdot p\,[\mathrm{GeV/c}] \quad\Longrightarrow\quad B\,[\mathrm{T}] = \frac{1}{0.29979}\cdot p\,[\mathrm{GeV/c}]\cdot\frac{1}{\rho\,[\mathrm{m}]} = 1.334\ \mathrm{T}$$

c) The earth's magnetic field will deflect the particles in the beam if it has a non-zero transverse (i.e. vertical or radial) component with respect to the particle's orbit. However, the strength of the earth's magnetic field (estimated to be $3\cdot 10^{-4}$ T) is far less than the bending magnet strength which is roughly 1 T. The tolerance of the magnets will be roughly 1% and therefore there will be corrector coils capable of correcting deviations of this order. Thus, we do not have to worry about the earth's magnetic field.

d)

$$F_{grav} = m_e g = 9.109\cdot 10^{-31}\ \mathrm{kg}\ \cdot\ 9.81\ \mathrm{m/s^2} \approx 8.94\cdot 10^{-30}\ \mathrm{N}$$
$$F_{bend} = evB = e\beta cB \approx ecB \approx 1.602\cdot 10^{-19}\ \mathrm{C}\ \cdot\ 3\cdot 10^{8}\ \mathrm{m/s}\cdot 1.334\ \mathrm{T} \approx 6.41\cdot 10^{-11}\ \mathrm{N}$$
$$\Longrightarrow F_{bend} \approx 10^{19}\cdot F_{grav}$$

3. Dipole Magnets vs. Static Electric Fields

Highly relativistic electrons: $v \approx c$. Recall the Lorentz force in absolute values: $F = q\,(E + vB)$. Using a static electric field: $F_E = qE$, but using a dipole magnet: $F_B = qvB \approx qcB$. By using dipole magnets the force applied to the particle is scaled by its velocity, thus much less field strength is required to apply the same force as a static electric field.

4. Hill's Equation

a) Hill's equation is:

$$x'' + k(s)\cdot x = 0$$

Here we assume $k(s) = k = const$:

$$x'' + kx = 0 \quad\Longrightarrow\quad x'' = -kx$$

This is the ordinary differential equation (ODE) of a harmonic oscillator (HO).

b) For simplicity we introduce the notation $x(0) = x_0$ and $x'(0) = x'_0$. The general (real) solution of the ODE $x'' = -kx$ is given by:

$$x(s) = x_0 \cdot \cos\sqrt{k}s \ + \ \frac{x'_0}{\sqrt{k}} \cdot \sin\sqrt{k}s$$

This is a harmonic oscillation for $\mathbf{k > 0}$.
If $\mathbf{k < 0}$ we make use of the identity $\sqrt{k} = i\sqrt{|k|}$:

$$x(s) = x_0 \cdot \cosh\sqrt{|k|}s \ + \ \frac{x'_0}{\sqrt{|k|}} \cdot \sinh\sqrt{|k|}s$$

This is exponential damping or excitation.
If $\mathbf{k = 0}$ the ODE becomes $x'' = 0$ with the general solution:

$$x(s) = x_0 + x'_0 s$$

This is linear propagation.

c) Within each interval i where $k(s) = k_i$ is constant the solution of Hill's equation is the solution of a harmonic oscillator with $k = k_i$. If the solving functions $x_i(s)$ are combined to a smooth function $x(s)$ one has an approximation for the solution of the original Hill's equation.

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