Prof. Dr. Albin F. Wrulich EPFL, Winter Term 2005/2006

Introduction to Particle Accelerator Physics

Tutorial 3 - Solutions

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1. Dispersion Function
From the lecture recall the general solution of the dispersion function

$$D(s) = D_0 \cos\frac{s}{\rho} + D_0'\rho \sin\frac{s}{\rho} + \rho\left(1 - \cos\frac{s}{\rho}\right)$$

We can rewrite this and do a Taylor expansion of the sine and cosine functions keeping our limit $\rho \to \infty$ in mind

$$\begin{aligned} D(s) &= D_0 \cos\frac{s}{\rho} + \rho - \rho\cos\frac{s}{\rho} + D_0'\rho\sin\frac{s}{\rho} \\ &= D_0\left(1 - \frac{s^2}{2\rho^2} + - \dots\right) + \rho\left(1 - 1 + \frac{s^2}{2\rho^2} - + \dots\right) + D_0'\rho\left(\frac{s}{\rho} - \frac{s^3}{3!\cdot\rho^3} + - \dots\right) \\ &= D_0\left(1 - \mathcal{O}(\rho^{-2})\right) + \mathcal{O}(\rho^{-1}) + D_0's + D_0'\mathcal{O}(\rho^{-2}) \end{aligned}$$

And finally for a straight section we apply the limit $\rho \to \infty$

$$\lim_{\rho\to\infty} D(s) = D_0 + D_0's$$

So if we enter a straight section with no dispersion at all ($D_0 = D_0' = 0$), $D(s)$ will remain zero. In order to generate dispersion, we have to include a bend in our accelerator optics where $\rho \neq 0$.

2. Momentum Compaction Factor
From the lecture recall the definition of the momentum compaction factor

$$\frac{\Delta L}{L} = \alpha_c \cdot \frac{\Delta p}{p}$$

From this we can derive the maximum absolute change in path length

$$\Delta L = \alpha_c \cdot \frac{\Delta p}{p} \cdot L = \alpha_c \cdot \frac{\Delta p}{p} \cdot \frac{c}{f} = 6.3 \cdot 10^{-4} \cdot 9 \cdot 10^{-4} \cdot \frac{3 \cdot 10^8 \text{ m/s}}{1.04167 \cdot 10^6 \text{ Hz}} = 163\ \mu\text{m}$$

Compared to the circumference of 288m this is a very small path length variation.

3. Stability Criterion for a Circular Accelerator
a)

$$M_{Rev} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

Here we assume $\alpha_0 = -\frac{\beta_0'}{2} = 0$ which gives $\gamma_0 = -1/\beta_0$ and thus

$$M_{Rev} = \begin{pmatrix} \cos\mu & \beta_0 \sin\mu \\ -\frac{1}{\beta_0}\sin\mu & \cos\mu \end{pmatrix}$$

b) The betatron tune is defined as follows

$$Q = \frac{\mu}{2\pi} = \frac{\phi_{Rev}}{2\pi}$$

where the total phase advance in one revolution (accelerator circumference C) is given by

$$\phi_{Rev} = \oint \frac{1}{\beta(s)}\, ds = \int_{s_0}^{s_0+C} \frac{1}{\beta(s)}\, ds$$

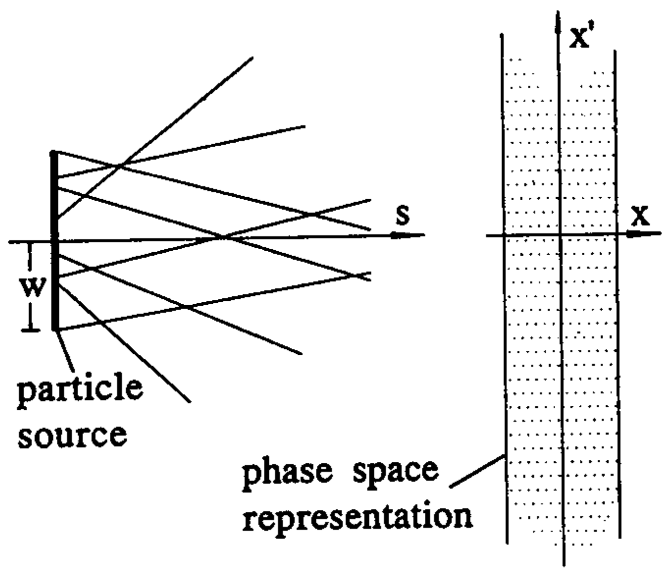
M_{Rev} looks like a rotation matrix for the rotation angle μ. The slight difference is the factor β_0 respectively $1/\beta_0$. This leads to an ellipse in phase space. A point on this ellipse advances by the angle μ for one revolution in the accelerator. The tune Q is the total betatron phase advance μ divided by 2π. Therefore Q is the number of betatron oscillations per revolution.

c) We can easily calculate the trace $Tr(M_{Rev}) = 2\cos\mu$ and since we know that the cos function has the co-domain [-1,+1] in $\mathbb{R}$ we can derive a simple stability criterion

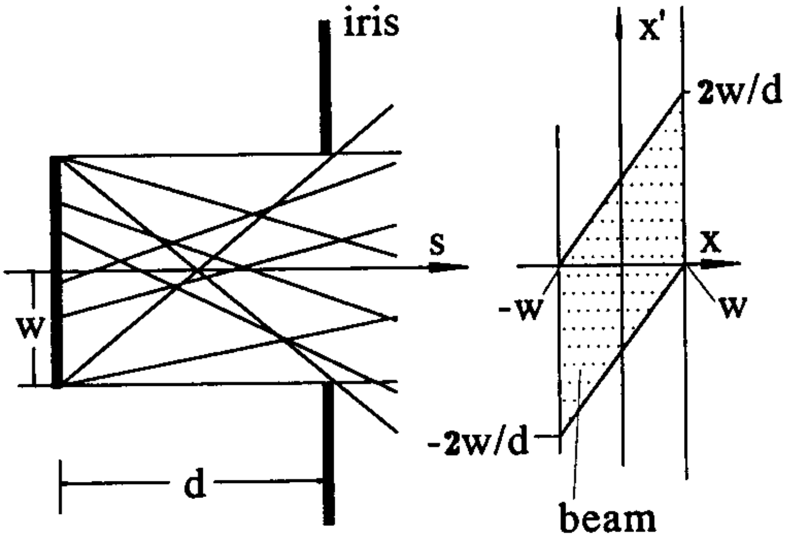
$$-1 \leq \frac{Tr(M_{Rev})}{2} \leq +1$$

4. Phase Space Representations of Particle Sources

a) Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\phi \in [-\pi/2, +\pi/2]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



b) Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.



5. Decapolar Magnetic Field

We chose the ansatz presented in the lecture

$$G_y(x) = \hat{g}x^4$$

where we have used a generic term proprtional to the gradient

$$\hat{g} \propto \frac{\mathrm{d}^4 G_y(x)}{\mathrm{d}x^4}$$

Since we know

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\mathrm{d}^2 G_y(x)}{\mathrm{d}x^2} y + \frac{\mathrm{d}f(y)}{\mathrm{d}y} = 0$$

we can derive $f(y)$

$$\begin{aligned} \nabla^2\phi = \frac{\mathrm{d}^2 G_y(x)}{\mathrm{d}x^2} y + \frac{\mathrm{d}f(y)}{\mathrm{d}y} &= 0 \\ 12\hat{g}x^2 y + \frac{\mathrm{d}f(y)}{\mathrm{d}y} &= 0 \\ \frac{\mathrm{d}f(y)}{\mathrm{d}y} &= -12\hat{g}x^2 y \\ \Longrightarrow f(y) &= -6\hat{g}x^2y^2 \end{aligned}$$

The potential is then

$$\begin{aligned} \phi(x,y) &= G_y(x)\ y + \int f(y)\mathrm{d}y \\ &= \hat{g}x^4 y - 2\hat{g}x^2y^3 \end{aligned}$$

The equipotential lines are given by

$$x^4 y - 2x^2 y^3 = \text{const}$$

The magnetic induction is given by

$$\vec{B}(x,y) = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix} = \begin{pmatrix} 4\hat{g}x^3 y - 4\hat{g}xy^3 \\ \hat{g}x^4 - 6\hat{g}x^2 y^2 \end{pmatrix}$$

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