Prof. Dr. Albin F. Wrulich EPFL, Winter Term 2005/2006

Introduction to Particle Accelerator Physics

Tutorial 4 - Solutions

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1. Quadrupole Errors and Tune Shifts

From the lecture recall the one-turn matrix at an arbitrary location:

$$M = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\gamma \sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix}$$

Assume now that at this location a very small gradient error is applied to the otherwise undisturbed optics (denoted by subscript 0):

$$\begin{aligned} \hat{M} &= \begin{pmatrix} 1 & 0 \\ -\Delta(kl) & 1 \end{pmatrix} \cdot M_0 \\ &= \begin{pmatrix} 1 & 0 \\ -\Delta(kl) & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\pi Q_0 + \alpha_0 \sin 2\pi Q_0 & \beta_0 \sin 2\pi Q_0 \\ -\gamma_0 \sin 2\pi Q_0 & \cos 2\pi Q_0 - \alpha_0 \sin 2\pi Q_0 \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\pi Q_0 + \alpha_0 \sin 2\pi Q_0 & \beta_0 \sin 2\pi Q_0 \\ -\gamma_0 \sin 2\pi Q_0 - \Delta(kl) \cos 2\pi Q_0 - \Delta(kl)\alpha_0 \sin 2\pi Q_0 & \cos 2\pi Q_0 - \alpha_0 \sin 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0 \end{pmatrix} \end{aligned}$$

In order to investigate the new tune $Q = Q_0 + \Delta Q$ we will compare the traces of the matrices:

$$\begin{aligned} Tr(M) &= Tr(\hat{M}) \\ 2\cos 2\pi Q &= 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0 \end{aligned}$$

We keep in mind that $Q = Q_0 + \Delta Q$ and make use of a trigonometric identity to rewrite the left hand side:

$$2\cos 2\pi Q_0 \cos 2\pi\Delta Q - 2\sin 2\pi Q_0 \sin 2\pi \Delta Q = 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0$$

We recall the assumption that the tune shift will be small $\Delta Q \ll 1$ which allows us to apply the two Taylor approximations $\cos 2\pi\Delta Q \approx 1$ and $\sin 2\pi\Delta Q \approx 2\pi\Delta Q$:

$$2\cos 2\pi Q_0 - 2\pi\Delta Q \, 2\sin 2\pi Q_0 = 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0$$

Which then gives us:

$$4\pi\Delta Q \sin 2\pi Q_0 = \Delta(kl)\beta_0 \sin 2\pi Q_0$$

Resulting in the tune shift:

$$\Delta Q = \frac{1}{4\pi}\beta_0 \Delta(kl)$$

2. Momentum Compaction and Transition Energy

From the lecture recall the definition of the momentum compaction factor:

$$\frac{\Delta L}{L} = \alpha_c \cdot \frac{\Delta p}{p}$$

In order to look at changes in period length ΔT we have to keep in mind how T and L are related and make use of the logarithmic derivative:

$$\begin{aligned} T &= \frac{L}{c\beta} \\ \log T &= \log L - \log c\beta \\ \Longrightarrow \frac{dT}{T} &= \frac{dL}{L} - \frac{d\beta}{\beta} \end{aligned}$$

In order to plug this together with the definition of the momentum compaction factor, we need to investigate $\frac{d\beta}{\beta}$:

$$\begin{aligned} p &= m_0\gamma\beta c \\ \frac{dp}{d\beta} &= m_0 c \frac{d}{d\beta}(\gamma\beta) \\ &= m_0 c\gamma + m_0 c\beta \frac{d\gamma}{d\beta} \\ &= m_0 c\gamma \left(1 + \beta^2\gamma^2\right) \\ &= m_0 c\gamma^3 \\ \Longrightarrow \frac{dp}{p} &= \gamma^2 \frac{d\beta}{\beta} \end{aligned}$$

We can now put together the two intermediate results and insert the definition of the momentum compaction factor:

$$\begin{aligned} \frac{\Delta T}{T} &= \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta} \\ &= \alpha_c \cdot \frac{\Delta p}{p} - \frac{1}{\gamma^2} \cdot \frac{\Delta p}{p} \\ &= \left(\alpha_c - \frac{1}{\gamma^2}\right) \frac{\Delta p}{p} \end{aligned}$$

This result shows how the revolution period changes with momentum. There is a special energy, the so-called *transition energy* γ_{tr}, defined as:

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}}$$

At transition energy the revolution period becomes independent of the momentum spread and stays constant for off-momentum particles.

3. Quadrupole Scan for Emittance Measurement

Assume a quadrupole with focussing strength kl where the tunable strength is given by k. Assume the drift distance to the screen monitor is given by L. The transfer matrix for this setup is then

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ kl & 1 \end{pmatrix} = \begin{pmatrix} 1 + Lkl & L \\ kl & 1 \end{pmatrix}$$

Recall the transformation properties of the Twiss parameters

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

where C denotes the cosine-like function and S denotes the sine-like function in the transfer matrix

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

Keeping this in mind we can write for β

$$\beta = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

Since we are measuring the beam profile on the screen monitor, we are actually interested in expressing σ_x

$$\begin{aligned} \sigma_x^2 = \varepsilon\beta &= C^2\varepsilon\beta_0 - 2SC\varepsilon\alpha_0 + S^2\varepsilon\gamma_0 \\ &= k^2 \cdot (L^2 l^2 \varepsilon\beta_0) + k \cdot (2Ll\varepsilon\beta_0 - 2L^2 l\varepsilon\alpha_0) + \varepsilon\beta_0 - 2L\varepsilon\alpha_0 + L^2\varepsilon\gamma_0 \\ &= k^2 c_2 + kc_1 + c_0 \end{aligned}$$

This is a parabolic expression in k. If we take data for σ_x^2 as a function of k we can derive the three coefficients c_2, c_1 and c_0 from a fit performed on the data. This allows us to express the initial Twiss parameters as functions of the fit values

$$\begin{aligned} \varepsilon\beta_0 &= \frac{1}{L^2}\frac{c_2}{l^2} \\ \varepsilon\alpha_0 &= \frac{1}{L^2}\left(\frac{c_2}{Ll^2} - \frac{c_1}{2l}\right) \\ \varepsilon\gamma_0 &= \frac{1}{L^2}\left(\frac{c_2}{L^2l^2} - \frac{c_1}{Ll} + c_0\right) \end{aligned}$$

We now recall that $\beta_0\gamma_0 - \alpha_0^2 = 1$ which allows us to calculate the emittance as a function of the fit values

$$\begin{aligned} \varepsilon^2 = \varepsilon^2(\beta_0\gamma_0 - \alpha_0^2) &= \varepsilon\beta_0 \cdot \varepsilon\gamma_0 - (\varepsilon\alpha_0)^2 \\ &= \frac{1}{L^4}\frac{c_2}{l^2}\left(\frac{c_2}{Ll^2} - \frac{c_1}{2l}\right) - \frac{1}{L^4}\left(\frac{c_2}{L^2l^2} - \frac{c_1}{Ll} + c_0\right)^2 \\ &= \frac{1}{4L^4l^2}\left(4c_0c_2 - c_1^2\right) \end{aligned}$$

Assistant: Simon C. Leemann, simon.leemann@psi.ch, 056/310 3343